Convex Equations and Differential Inclusions in Hybrid Systems

D.A. van Beek, A. Pogromsky, H. Nijmeijer, J.E. Rooda

***Abstract*— Differential equations with discontinuous right hand sides enable modeling and analysis of control systems with switching elements at a high level of abstraction. Solutions of these differential equations are based on the Filippov, Utkin or similar solution concepts. These solution concepts are in general inconvenient for modeling and verification using formal languages, because they lead to ambiguities in differential algebraic equations. This paper introduces convex equations to avoid such ambiguities in formal languages. Convex equations integrate the functionality of the Filippov solution concept with much of the Utkin solution concept in general differential algebraic equations. A formal semantics of convex equations is given, and an example model is specified using a combined discrete-event / continuous-time formalism.**

I. Introduction

Hybrid systems related research is based on two, originally different, world views: on the one hand the dynamics and control world view, and on the other hand the computer science world view.

The dynamics and control world view is that of a predominantly continuous-time (CT) system, which is modeled by means of differential (algebraic) equations (DAEs), or by means of a set of trajectories. Hybrid phenomena are modeled by means of discontinuous functions and/or switched equation systems.

The computer science world view is that of a predominantly discrete-event (DE) system. There exist many different languages for modeling of discrete-event systems. A well-known model is a finite-state automaton, but modeling of DE systems is also based on, among others, process algebra, Petri nets, and data flow languages. For modeling of hybrid phenomena, different DE languages are extended in different ways with some form of differential (algebraic) equations.

For analysis of hybrid systems in the dynamics and control domain, notations and concepts from mathematics are used. For analysis of hybrid systems in the computer science domain, however, generally used notations and concepts from mathematics are not sufficient. To enable analysis of DE or hybrid systems in the computer science domain, the notations (syntax), and their exact meaning (semantics) need to be defined, leading thus to *formal* languages.

For the analysis of hybrid systems, it can be beneficiary to integrate concepts from dynamics and control with concepts from computer science. This is for example the case in embedded system design, where an embedded computer is used to control a physical system. Dynamics and control theory can be used to analyse such systems under the assumption that the embedded computer has zero response time, or at least a fixed response time. In reality, however, response times of real-time operating systems need not be fixed. Such real-time control systems can be approximated by means of DE models with stochastic response times, leading to combined DE/CT models.

This paper focusses on the incorporation of 'differential (algebraic) equations with discontinuous right hand sides', from the dynamics and control domain, into formal hybrid languages from the computer science domain. Such differential equations allow modeling of relays, valves or any kind of on/off switching elements at a high level of abstraction in control systems with so called sliding modes. The χ (Chi) language is well suited to modeling such phenomena, since it allows discontinuous functions in differential algebraic equations, unlike for example hybrid automata considered in [1]. Originally, χ was a modeling and simulation language [2] suited to (stochastic) discrete-event systems and hybrid systems, including the associated discrete-event control systems. Later, a formal semantics for discrete-event χ [3] and hybrid χ [4] was developed for the purpose of verification.

Section 2 presents a background on solution concepts, Section 3 shows how the use of different solution concepts can lead to ambiguous specifications. To avoid this ambiguity, a new 'convex equality' operator is proposed in Section 4, while Section 5 explores other means to obtain unambiguous specifications. Section 6 presents a formal semantics for general systems of differential algebraic equations in combination with the convex equality operator, and defines some properties of this operator. An example is presented in Section 7, and Section 8 presents the conclusions.

II. Background on solution concepts

One of the most popular (and simplest) solution concepts for discontinuous differential equations is attributed to Filippov [5]. To describe his concept consider the following system

$$\dot{x} = f(t, x), \quad x \in \mathbb{R}^n, \quad t \in [t_0, \infty) \qquad (1)$$

and suppose that the function $f(t, x)$ is piece-wise continuous in the domain of interest G, has discontinuities only on a set of zero measure and is bounded on the region G. At each point (t, x) of the domain $G \subset [t_0, \infty) \times \mathbb{R}^n$ we construct a set $F(t, x)$ which consists of only one point $f(t, x)$ if the function f is continuous at this point, otherwise the set $F(t, x)$ is defined as the smallest convex closed set containing all limit points of the function $f(t, x^*)$, $x^* \to x$,

This work was partially supported by the EU projects SICONOS (IST-2001-37172) and HyCon (FP6-2003-IST-2). All authors are with the Dept. of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands {d.a.v.beek, a.pogromsky, h.nijmeijer, j.e.rooda}@tue.nl

t is constant. With such a set $F(t,x)$ a Filippov solution to system (1) is a solution of the following *differential inclusion*:

$$\dot{x} \in F(t,x) \tag{2}$$

There is a rather general way to construct a solution concept for discontinuous systems. Here we briefly make an overview of this method following [5], see also [6].

Consider a system

$$\dot{x} = f(t,x,u_1(t,x),\dots,u_r(t,x)) \tag{3}$$

where $x \in \mathbb{R}^n$, the vector-valued function f is continuous in the set of arguments, and the scalar or vector-valued functions u_i, $i = 1,\dots,r$ are discontinuous on the respective sets M_i. At each point of discontinuity of the function u_i, a closed set $U_i(t,x)$ must be given, which is a set of possible values of the argument u_i of the function $f(t,x,u_1,\dots,u_r)$. For $i \neq j$, the arguments u_i and u_j are supposed to vary independently of one another on the sets $U_i(t,x)$ and $U_j(t,x)$, respectively. At the points where the function u_i is continuous, the set $U_i(t,x)$ consists of one point $u_i(t,x)$. At the points of discontinuity of the function u_i the set $U_i(t,x)$ contains all limit points for any sequence of the form $v_{ik} \in U_i(t_k,x_k)$, where $t_k \to t$, $x_k \to x$, $k = 1,2,\dots$. The set $U_i(t,x)$ is required to be convex. Then the solution of the differential inclusion

$$\dot{x} \in F(t,x,U_1(t,x),\dots,U_r(t,x)) \tag{4}$$

can be considered as a solution to the system of differential equations (3). With a similar approach one can define Utkin's equivalent control solution [7] and other solution concepts (see, e.g. [5]). The Utkin equivalent control definition applies to systems of the form (3), where f is a continuous vector-valued function, $u_i(t,x)$ is a scalar function discontinuous only on a smooth surface $S_i(\varphi_i(x) = 0)$, $i = 1,\dots,r$. At points belonging to one surface (or intersection of m surfaces) one assumes that

$$\dot{x} = f(t,x,u_1^{eq}(t,x),\dots,u_m^{eq}(t,x), u_{m+1}(t,x),\dots,u_r(t,x)), \tag{5}$$

where *equivalent controls* u_j^{eq}, $j = 1,\dots,m$ are defined so that the vector field f is tangent to the surfaces $S_1,\dots,S_m$, and the value $u_i^{eq}(t,x)$ is contained in the closed interval $[u_i^-(t,x), u_i^+(t,x)]$. Here u_i^-, u_i^+ are the limit points of the function u_i on both sides of the surfaces S_i. This solution concept can be reduced to the differential inclusion (4), where $U_i(t,x)$ is a segment with end-points $u_i^-(t,x), u_i^+(t,x)$, and for those u_i which are continuous at the point (t,x), the set $U_i(t,x)$ consists of just one point $u_i(t,x)$.

III. Different solution concepts lead to ambiguous specifications

Consider the following differential equation as an introductory example

$$\dot{x} = -\operatorname{sign}(x) + 0.5 \tag{6}$$

where

$$\operatorname{sign}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

which has a solution in the Filippov sense $x(t) = 0$ for x initially equal to 0. Therefore, solutions according to the Filippov and Utkin solution concepts no longer satisfy the equations at each point of time t: $\dot{x}(t) \neq -\operatorname{sign}(x(t)) + 0.5$ for $x(t) = \dot{x}(t) = 0$, where $x(t)$ and $\dot{x}(t)$ denote the values of x and $\dot{x}$ at time-point t. The equality symbol '=' no longer means that the left and right hand sides need to have the same value. Another consequence of the Filippov and Utkin solution concepts is that the differential algebraic equation system

$$\begin{aligned} \dot{x}_1 &= -u_1 + \varepsilon(t) \\ \dot{x}_2 &= u_2 \\ u_1 &= \operatorname{sign}(x_1) \\ u_2 &= \operatorname{sign}(x_1) \end{aligned} \tag{7}$$

can have different meanings. Consider initial conditions $(x_1, x_2) = (0,0)$ and $|\varepsilon(t)| < 1$, where ε denotes some function of time. According to the Caratheodory solution concept, there is a solution only for $\varepsilon(t) = \operatorname{sign}(0) = 0$. The Filippov solution concept does not define a solution for (7). However, if (7) is rewritten in the form

$$\begin{aligned} \dot{x}_1 &= -\operatorname{sign}(x_1) + \varepsilon(t) \\ \dot{x}_2 &= \operatorname{sign}(x_1), \end{aligned} \tag{8}$$

the Filippov solution of (7) can informally be defined as the solution of (8).

To find the Filippov solution for this system we have to find a set of limit points of the right hand side and the corresponding differential inclusion. It is not difficult to see that in this case the set of points $F(t,0)$ is defined by

$$F(t,0) = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xi + \begin{pmatrix} \varepsilon(t) \\ 0 \end{pmatrix} \mid \xi \in [-1,1] \right\} \tag{9}$$

The corresponding differential inclusion

$$\dot{x} \in F(t,x)$$

has a unique solution for arbitrary initial points. Particularly, for x_1 and x_2 initially 0 the corresponding solution is

$$x_1(t) = 0, \quad x_2(t) = \int_0^t \varepsilon(s)ds$$

Consider now the Utkin solutions of (7). For this model at the discontinuity point $x_1 = 0$, the vector $\dot{x}$ can take values from the set

$$\left\{ \begin{pmatrix} -\xi_1 + \varepsilon(t) \\ \xi_2 \end{pmatrix} \mid \xi_1 \in [-1,1], \xi_2 \in [-1,1] \right\}, \tag{10}$$

which is larger than (9). Therefore, an Utkin solution for (7) can be different from a Filippov solution for (8), which can be informally understood as a Filippov solution for (7). More precisely, the Utkin solution concept allows multiple solutions for (7) while the Filippov solution concept allows only a unique solution for (8), for all initial points.

It appears that the current mathematical notation for DAEs is not sufficiently expressive to avoid ambiguity. Although for experts in the field of DAEs this may in general not be a problem, for non-experts, this is at least confusing. For tool support in the areas of computer-based analysis, and for formal languages, the problem is even clearer. Here, the notation *must* be unambiguous.

IV. Convex equations as a means for unambiguous specifications

In this paper we propose an unambiguous notation by introduction of a new binary mathematical operator $\mathrel{\mathring{=}}$ that denotes convex equality. Informally, the meaning of

$$v \mathrel{\mathring{=}} f(\dot{x}, x, u, t), \tag{11}$$

where v denotes some scalar or vector, e.g. $v = u_1$ or $v = \begin{pmatrix} \dot{x}_2 \\ u \end{pmatrix}$, and f denotes some scalar or vector valued function with the same dimension as v, is as follows. Convex equation (11) is satisfied in either of the following two cases: 1) for all points for which f is a continuous function if $v = f(\dot{x}, x, u, t)$, and 2) at each discontinuity point of f if v is an element of the convex combination of limit points of f in the discontinuity point.

Normal and convex equations are predicates. Depending on the values of the variables, they are either true or false. For example $1 = \text{sign}(0.2)$ and $1 \mathrel{\mathring{=}} \text{sign}(0.2)$ are both true, $0.2 = \text{sign}(0)$ is false, but $0.2 \mathrel{\mathring{=}} \text{sign}(0)$ is true.

As another example,

$$y \mathrel{\mathring{=}} \text{sign}(x) \tag{12}$$

is equivalent to

$$y = \text{sign}(x) \vee (x = 0 \wedge y \in [-1, 1]) \tag{13}$$

This means that whenever (12) occurs in a model, it can be replaced by (13) without changing the meaning of the model.

Where the system of equations (7) can have three different meanings, related to either the Caratheodory, the Filippov or the Utkin solution concept, the meaning of a model specified using convex equations is unambiguous. Using convex equations, equality '$=$' is indeed proper equality. This means that solutions of model (7) then correspond to the solutions according to the Caratheodory solution concept. The Utkin solution of (7) is defined by the model

$$\begin{aligned} \dot{x}_1 &= -u_1 + \varepsilon(t) \\ \dot{x}_2 &= u_2 \\ u_1 &\mathrel{\mathring{=}} \text{sign}(x_1) \\ u_2 &\mathrel{\mathring{=}} \text{sign}(x_1) \end{aligned} \tag{14}$$

which can be rewritten without convex equations as

$$\begin{aligned} \dot{x}_1 &= -u_1 + \varepsilon(t) \\ \dot{x}_2 &= u_2 \\ u_1 &= \text{sign}(x_1) \vee (x_1 = 0 \wedge u_1 \in [-1, 1]) \\ u_2 &= \text{sign}(x_1) \vee (x_1 = 0 \wedge u_2 \in [-1, 1]) \end{aligned} \tag{15}$$

Model (7) with respect to the Filippov solution of (8) can be unambiguously specified using the convex equality operator in several ways. The following two specifications have the same solutions for x_1 and x_2 as the Filippov solution for (8), but the models below also define solutions for u_1 and u_2:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ u_1 \\ u_2 \end{pmatrix} \mathrel{\mathring{=}} \begin{pmatrix} -u_1 + \varepsilon(t) \\ u_2 \\ \text{sign}(x_1) \\ \text{sign}(x_1) \end{pmatrix} \tag{16}$$

and

$$\begin{aligned} \dot{x}_1 &= -u_1 + \varepsilon(t) \\ \dot{x}_2 &= u_2 \\ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &\mathrel{\mathring{=}} \begin{pmatrix} \text{sign}(x_1) \\ \text{sign}(x_1) \end{pmatrix} \end{aligned} \tag{17}$$

When solutions for u_1 and u_2 are not considered relevant, the model can be simplified to

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \mathrel{\mathring{=}} \begin{pmatrix} -\text{sign}(x_1) + \varepsilon(t) \\ \text{sign}(x_1) \end{pmatrix} \tag{18}$$

Specification (17) has the following meaning:

$$\begin{aligned} &\dot{x}_1 = -u_1 + \varepsilon(t) \\ &\dot{x}_2 = u_2 \\ &\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \text{sign}(x_1) \\ \text{sign}(x_1) \end{pmatrix} \vee \left(x_1 = 0 \wedge \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \text{co}(L)\right), \end{aligned} \tag{19}$$

where

$$L = \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

denotes the set of limit points. The convex combination $\text{co}(L)$ is a line segment connecting the two limit points, so that $u_1 = u_2$. Therefore the above specification is equivalent to

$$\begin{aligned} \dot{x}_1 &= -u_1 + \varepsilon(t) \\ \dot{x}_2 &= u_2 \\ u_1 &= u_2 \\ u_2 &\mathrel{\mathring{=}} \text{sign}(x_1) \end{aligned} \tag{20}$$

The most important advantage of the new convex equality operator is that the meaning of a system of DAEs and convex equations is straightforward and unambiguous. A second advantage of convex equations is that they can be used in general (implicit) DAEs and integrate the functionality of these DAEs with the Filippov solution concept, and with much, but not all, of the Utkin solution concept. This, in principle, enables formal (computer-based) analysis of a wide range of hybrid models.

V. Other means for unambiguous specifications

This section explores other means to obtain unambiguous specifications. We first consider the possibility of adopting the most widely used Filippov solution concept as *the* solution concept. In this way, the semantics of equations in a language with associated computer-aided analysis tools could be defined in the Filippov sense. This is a valid option if only ordinary differential equations $\dot{x} = f(x, t)$ (ODEs) are considered. For such systems of equations, the Filippov solution concept unambiguously defines the meaning. In many cases, however, ODEs are not sufficiently expressive, and DAEs are required. Consider for example the friction phenomenon. A driving force F_{d} is applied to a body on a flat surface with frictional force F_{f} (see Fig. 1). When

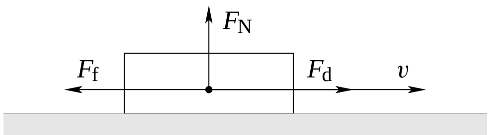

Fig. 1. Body with friction.

the body is moving with velocity v, the frictional force is given by $F_{\mathrm{f}} = \mu F_{\mathrm{N}}$, where $F_{\mathrm{N}} = mg$, and μ is the friction coefficient. When the velocity of the body is zero and $|F_{\mathrm{d}}| \leq \mu F_{\mathrm{N}}$, the frictional force neutralizes the applied driving force. If we assume $\mu F_{\mathrm{N}} = 1$, $m = 1$, $F_{\mathrm{d}} = 0.5$, and introduce a variable y that indicates whether x is positive, negative or zero, the model becomes:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= 0.5 - F_{\mathrm{f}} \\ F_{\mathrm{f}} &= \mathrm{sign}(v) \\ y &= \mathrm{sign}(x) \end{aligned} \tag{21}$$

The Filippov solution concept defines the solution for x and v of (21) as the solution of

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ 0.5 - \mathrm{sign}(v) \end{pmatrix}$$

in the sense of (2). If initially x and v equal 0, the solution for x and v is $x(t) = v(t) = 0$. However, since the friction force F_{f} is a physically relevant variable, a solution of (21) should also define the value of F_{f} (and the value of y) as a function of time. The only way for model (21) to be consistent with the solution $x(t) = v(t) = 0$ is for $F_{\mathrm{f}} = 0.5$, which leads to the ambiguity that $F_{\mathrm{f}} = \mathrm{sign}(v) = \mathrm{sign}(0) = 0.5$, whereas $y = \mathrm{sign}(x) = \mathrm{sign}(0) = 0$. Therefore, although the Filippov solution concept itself is unambiguously defined for ODEs, its application to systems of differential *algebraic* equations may lead to ambiguities.

Another attempt to avoid ambiguities is to explicitly introduce set valued functions in the language. This would lead to, for example:

$$\dot{x} \in -\mathrm{Sign}(x) + \varepsilon$$

where the set valued function Sign could be defined as:

$$\mathrm{Sign}(x) = \begin{cases} \{-1\} & x < 0 \\ [-1, 1] & x = 0 \\ \{1\} & x > 0 \end{cases}$$

A disadvantage of using set valued functions in formal models is that syntax / notations are needed to define set valued functions. This leads to, for example, to two different sign functions: $\mathrm{sign} \in \mathbb{R} \to \mathbb{R}$ and $\mathrm{Sign} \in \mathbb{R} \to \mathcal{P}(\mathbb{R})$. Furthermore, the meaning of expressions on set valued functions, such as f *op* g for arbitrary set valued functions f and g and numeric operator *op*, then also needs to be defined. As a conclusion we can state that although the Filippov solution concept, differential inclusions, and set-valued functions are convenient for mathematical analysis, they are inconvenient for modeling and computer aided analysis.

VI. Formal semantics of DAEs with convex equations

In the previous sections, the meaning of the convex equality operator and convex equations was defined informally. This section presents a formal semantics that unambiguously and rigourously defines the meaning of convex equations. This is especially important for formal analysis of hybrid systems where the discrete-event behavior consists of more than just switched equations, but includes, among others, assignments, communication, parallel and alternative composition (see for example [4]).

Differential algebraic equations are defined as predicates over variables $x, \dot{x}, y, t$, where the values of these variables can be defined as either (vectors of) reals ($x \in \mathbb{R}^k$) or (vector) functions from time to reals. Consider for example

$$\dot{x} = -x + 1 \tag{22}$$

When the values of variables are considered to be functions ($x, \dot{x} \in T \to \mathbb{R}^k$, $T = \mathbb{R}_{\geq 0}$), the left and right hand sides of (22) denote functions. This means that the 'number' 1 then denotes the constant function of time that is 1 for all arguments. Furthermore, $\dot{x}$ then denotes the function $\frac{d}{dt}x$, or function $\dot{x}$ is implicitly defined as $x(t) = \int_0^t \dot{x}(s)ds$.

The semantics in this section considers the values of variables to be (vectors of) reals ($x \in \mathbb{R}^k$), instead of (vector) functions from time to reals. Therefore, solutions $X, \dot{X} \in T \to \mathbb{R}^k$ are defined, such that the function values satisfy predicate (22) at all points of time: $\forall_{t \in T}\ \dot{X}(t) = -X(t) + 1$, and $X(t) = \int_0^t \dot{X}(s)ds$. Furthermore, in the formal semantics a clear distinction is made between variable names and their values. The essence of this formal semantics is a separation between:

- The semantics of a single predicate over variables and dotted variables (derivatives). This semantics is defined in terms of functions of time for all variables. Predicates can denote DAEs or convex equations.
- The semantics of a system of several predicates in terms of the semantics of its individual predicates.
- The semantics of the convex equality operator $\mathrel{\overset{\circ}{=}}$. This semantics defines for which values of its variables a convex equation (predicate) is satisfied.

A. Equation semantics

Consider a set of variable names $V = \{x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, y_1, \ldots, y_m, t\}$, for which we use the following abbreviation $V = \{x, \dot{x}, y, t\}$. Variable name t denotes time. Variables $x_i, \dot{x}_i, y_j$ take values in $\mathbb{R}$

$(1 \le i \le n, 1 \le j \le m)$, and t takes values in T. The values of these variables is captured by means of a *valuation* $\sigma \in \Sigma$, where $\Sigma = V \to (\mathbb{R} \cup \perp)$, where $\perp$ denotes an undefined 'value'.

An equation $q \in Q$, where Q denotes the set of all differential algebraic equations over variables from V, is a predicate over variables from V. Two types of equations q are considered: normal equations $e = e'$, and convex equations $e \mathrel{\mathring{=}} e'$, where $e, e' \in E^k$, and E^k denotes the set of k-tuples of real-valued expressions. Elements of E^k are syntactically represented as k-tuples or as $k \times 1$ matrices. E.g. $\begin{pmatrix} \dot{x} - y \\ 2x \end{pmatrix} \in E^2$ and $(\dot{x} - y, 2x) \in E^2$ denote the same 2-tuple.

Variables are evaluated by means of valuation function σ, expressions and equations are evaluated by means of evaluation function $\text{eval} \in ((E^k \cup Q) \times \Sigma) \to (\mathbb{R}^k \cup \{\text{true}, \text{false}\})$.

We also define a second evaluation function $\text{eval}' \in ((E^k \cup Q) \times \mathbb{R}^{2n+m+1}) \to (\mathbb{R}^k \cup \{\text{true}, \text{false}\})$, such that

$$\begin{aligned} \text{eval}'(\xi, c) = \text{eval}(\xi, \{&x_1 \mapsto c_1, \ \ldots, \ x_n \mapsto c_n, \\ &\dot{x}_1 \mapsto c_{n+1}, \ \ldots, \ \dot{x}_n \mapsto c_{2n}, \\ &y_1 \mapsto c_{2n+1}, \ \ldots, \ y_m \mapsto c_{2n+m}, \ t \mapsto c_{2n+m+1}\}), \end{aligned}$$

where ξ denotes either an expression or an equation, and $c = (c_1, \ldots, c_{2n+m+1}) \in \mathbb{R}^{2n+m+1}$ is called a *data point*. Furthermore, we define $\sigma(x), \sigma(\dot{x}), \sigma(y)$ to denote the tuples $(\sigma(x_1), \ldots, \sigma(x_n))$, $(\sigma(\dot{x}_1), \ldots, \sigma(\dot{x}_n))$, $(\sigma(y_1), \ldots, \sigma(y_m))$, respectively.

Consider for example the set of variables $V = \{x_1, \dot{x_1}, y_1, t\}$, the valuation $\sigma = \{x_1 \mapsto 1, \dot{x_1} \mapsto 6, y_1 \mapsto 4, t \mapsto 1\}$, and data point $c = (1, 6, 4, 1)$. Then $\sigma(x_1) = 1$, $\sigma(\dot{x}) = 6$, $\text{eval}(x_1 + y_1 + t, \sigma) = \text{eval}'(x_1 + y_1 + t, c) = 6$, $\text{eval}(\dot{x}_1 = x_1 + y_1 + t, \sigma) = \text{eval}'(\dot{x}_1 = x_1 + y_1 + t, c) = \text{true}$, and $\text{eval}(\dot{x}_1 = x_1 + y_1, \sigma) = \text{eval}'(\dot{x}_1 = x_1 + y_1, c) = \text{false}$.

The semantics of an equation q with valuation σ (where $\sigma(\dot{x})$ and $\sigma(y)$ may be undefined) is defined in terms of a set of solutions on interval $[t_0, t_1]$, $t_0 = \sigma(t)$, $t_1 > t_0$. A solution is a pair (X, Y), where X denotes a solution function for variables x and Y denotes a solution function for variables y. The set of solutions is defined by means of solution function $\mathcal{S} \in (\Sigma \times Q \times T) \to \mathcal{P}((T \to \mathbb{R}^n) \times (T \to \mathbb{R}^m))$ such that

$$\begin{aligned} &\forall (X, Y) \in \mathcal{S}(\sigma, q, t_1) : X(t_0) = \sigma(x) \ \wedge \\ &\quad (\exists \dot{X} \in T \to \mathbb{R}^n : (\forall \tau \in [t_0, t_1] : X(\tau) = \int_{t_0}^{t_1} \dot{X}(s) ds \\ &\qquad \wedge (\dot{X}(\tau), X(\tau), Y(\tau), \tau) \models q \ )), \end{aligned} \tag{23}$$

where $c \models q$, $c \in \mathbb{R}^{2n+m+1}$, means that equation q is satisfied for data point c. Therefore, $c \models q$ if and only if $\text{eval}'(q, c) = \text{true}$. Furthermore, the resulting valuation σ' is defined as: $\sigma'(x) = X(t_1)$, $\sigma'(y) = Y(t_1)$, and $\sigma'(t) = t_1$.

The set of solutions of a system of equations $q_1, \ldots, q_p$ is $\mathcal{S}_1 \cap \ldots \cap \mathcal{S}_p$, where $\mathcal{S}_i$ denotes the set of solutions of equation q_i $(i = 1, \ldots, p)$.

B. Convex equation semantics

We define a convex equation $e \mathrel{\mathring{=}} e'$, $(e, e' \in E^k)$, to be satisfied for data point $c \in \mathbb{R}^r$, $r = 2n + m + 1$, if and only if $\mathbf{0} \in \text{cl}(\text{co}(\mathcal{L}(\tilde{e} - \tilde{e}', c)))$, so that

$$\text{eval}'(e \mathrel{\mathring{=}} e', c) \quad = \quad \mathbf{0} \in \text{cl}(\text{co}(\mathcal{L}(\tilde{e} - \tilde{e}', c))), \tag{24}$$

where $\tilde{e} \in \mathbb{R}^r \to \mathbb{R}^k$ denotes the function obtained by evaluating expression e for values $s \in \mathbb{R}^r$: $\tilde{e}(s) = \text{eval}'(e, s)$. Furthermore, $\mathcal{L}(f, c)$ denotes the set of limit points of function f in point c, $\text{co}(L)$ denotes the smallest convex set containing L, $\text{co} \in \mathcal{P}(\mathbb{R}^k) \to \mathcal{P}(\mathbb{R}^k)$, and $\text{cl}(L)$ denotes the closure of set L, $\text{cl} \in \mathcal{P}(\mathbb{R}^k) \to \mathcal{P}(\mathbb{R}^k)$.

Informally, without making a distinction between variable names and their values, the semantics of convex equations can be defined in an easier way. Informally, a convex equation can be written as $g(\dot{x}, x, y, t) \mathrel{\mathring{=}} 0$, where $x \in \mathbb{R}^n$, $\dot{x} \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and time $t \in \mathbb{R}_{\ge 0}$. Such a convex equation is satisfied in point $c \in \mathbb{R}^{2n+m+1}$ if and only if

$$\mathbf{0} \in \text{cl}(\text{co}(\mathcal{L}(g, c)))$$

C. Limit points

Function $\mathcal{L} \in ((\mathbb{R}^r \to \mathbb{R}^k) \times \mathbb{R}^r) \to \mathcal{P}(\mathbb{R}^k)$ is defined as:

$$\begin{aligned} &\mathcal{L}(f, c) = \\ &\quad \{l \in \mathbb{R}^k \mid \exists \{c_i\} \in \Psi : (\lim_{i \to \infty} c_i = c, l = \lim_{i \to \infty} f(c_i)\}, \end{aligned}$$

where Ψ denotes the set of all infinite sequences of points from $\mathbb{R}^r$, and $\{c_j\}$ denotes the infinite sequence $c_0, \ldots, c_j, c_{j+1}, \ldots$, such that $c_j \in \mathbb{R}^r$ for all j.

A set $A \in \mathcal{P}(\mathbb{R}^k)$ is called convex if $\forall a, b \in A : \forall \alpha \in [0, 1] : \alpha a + (1 - \alpha) b \in A$. A geometrical interpretation is that for any two points a and b of A, all points of a line-segment joining a and b also belong to A. For a finite set of limit points L, $\text{cl}(\text{co}(L)) = \text{co}(L)$, and $\text{co}(L)$ is defined as:

$$\begin{aligned} \text{co}(\{l_1, \ldots, l_k\}) = \{ \ &\alpha_1 l_1 + \ldots + \alpha_k l_k \mid \\ &\alpha_1, \ldots, \alpha_k \in \mathbb{R}_{\ge 0}, \ \alpha_1 + \ldots + \alpha_k = 1, \ k \ge 1 \ \} \end{aligned}$$

D. Properties of convex equations

Below three properties of the convex equality operator are defined:

$$c \models (v \mathrel{\mathring{=}} e) \ \text{ iff } \ \text{eval}'(v, c) \in \text{cl}(\text{co}(\mathcal{L}(\tilde{e}, c))) \tag{25}$$

$$\begin{aligned} &\tilde{e} - \tilde{e}' \text{ is continuous in } c \quad \Rightarrow \\ &\qquad\qquad (c \models (e \mathrel{\mathring{=}} e') \text{ iff } c \models (e = e')) \end{aligned} \tag{26}$$

where $e, e' \in E^k$, and v denotes a k-tuple of variables (e.g. for $k = 3$, v could denote $(\dot{x_1}, y_1, y_2)$, and for $k = 1$, v could denote $\dot{x_1}$).

$$\forall_{\sigma \in \Sigma, t_1 \in T} : \mathcal{S}(\sigma, \dot{x} \mathrel{\mathring{=}} e_{xt}, t_1) = \mathcal{S}_{\text{Filippov}}(\sigma, \dot{x} = e_{xt}, t_1) \tag{27}$$

where e_{xt} denotes an expression over x and t, such that $\tilde{e}_{xt}$ is continuous for t, $\mathcal{S}$ denotes the solution function defined in (23), and $\mathcal{S}_{\text{Filippov}}(\sigma, \dot{x} = e_{xt}, t_1)$ denotes the set of Filippov solutions of $\dot{x} = e_{xt}$ for initial state σ up to time-point t_1.

Property (25) makes it easier to understand the meaning of the usual form of convex equations $v \mathrel{\mathring{=}} e$, where v is some variable, and e some expression: $v \mathrel{\mathring{=}} e$ is true in data point c if and only if the value of v at data point c is an element of the closure of the convex combination of limit points of function $\tilde{e}$ at data point c.

Property (26) means that convex equality is the same as normal equality for values where the left hand expression minus the right hand expression represents a continuous function. Stated informally: $v \mathrel{\mathring{=}} f(x, \dot{x}, y, t)$, where v is some variable, has the same meaning as $v = f(x, \dot{x}, y, t)$ for all points where f is a continuous function.

Property (27) shows that convex equations are at least as expressive as normal equations using the Fillipov solution concept. Stated informally: the Fillipov solution concept is defined only for equations of the form $\dot{x} = f(x, t)$. Such equations are expressed by means of convex equations as $\dot{x} \mathrel{\mathring{=}} f(x, t)$, and the set of solutions is the same.

E. Proofs

To prove Property (25) we use Definition (24):

$$\begin{aligned}
&\mathrm{eval}'(v \mathrel{\mathring{=}} e, c)\\
&\quad = \mathbf{0} \in \mathrm{cl}(\mathrm{co}(\mathcal{L}(\tilde{v} - \tilde{e}, c)))\\
&\quad = \mathbf{0} \in \mathrm{cl}(\mathrm{co}(\mathrm{eval}'(v, c) - \mathcal{L}(\tilde{e}, c)))\\
&\quad = \mathbf{0} \in \mathrm{cl}(\mathrm{co}(\mathcal{L}(\tilde{e}, c) - \mathrm{eval}'(v, c)))\\
&\quad = \mathrm{eval}'(v, c) \in \mathrm{cl}(\mathrm{co}(\mathcal{L}(\tilde{e}, c)))
\end{aligned}$$

To prove Property (26), we need to prove that for all values c for which function $\tilde{e} - \tilde{e}'$ is continuous, if $\mathrm{eval}'(e = e', c)$ is true then $\mathrm{eval}'(e \mathrel{\mathring{=}} e', c)$ is true, and if $\mathrm{eval}'(e = e', c)$ is false then $\mathrm{eval}'(e \mathrel{\mathring{=}} e', c)$ is false.

If function $\tilde{e}$ is continuous in c then $\tilde{e}$ has one limit point only in c: $\mathcal{L}(\tilde{e}, c) = \{\mathrm{eval}'(e, c)\}$. Therefore, with definition (24) it follows that:

$$\begin{aligned}
&\mathrm{eval}(e \mathrel{\mathring{=}} e', c)\\
&\quad = \mathbf{0} \in \mathrm{cl}(\mathrm{co}(\mathcal{L}(\tilde{e} - \tilde{e}', c)))\\
&\quad = \mathbf{0} \in \mathrm{cl}(\mathrm{co}(\{\mathrm{eval}'(e - e', c)\}))\\
&\quad = \mathbf{0} \in \{\mathrm{eval}'(e - e', c)\}
\end{aligned}$$

If $\mathrm{eval}'(e = e', c)$ is true, then $\mathrm{eval}'(e - e', c) = \mathbf{0}$, so that $\mathrm{eval}'(e \mathrel{\mathring{=}} e', c)$ is true. If $\mathrm{eval}'(e = e', c)$ is false, $\mathrm{eval}'(e - e', c) \neq \mathbf{0}$, so that $\mathrm{eval}'(e \mathrel{\mathring{=}} e', c)$ is false, which concludes the proof.

We prove Property (27) in a somewhat informal way, since $\mathcal{S}_{\mathrm{Filippov}}$ is not formally defined in this paper. We first consider the points for which $\tilde{e}_{xt}$ is a continuous function. In such a case Property (26) together with (23) defines that $\mathcal{S}(\sigma, \dot{x} \mathrel{\mathring{=}} e_{xt}, t_1) = \mathcal{S}(\sigma, \dot{x} = e_{xt}, t_1)$ which corresponds to the Filippov solution. In points c for which $\tilde{e}_{xt}$ is discontinuous, Property (25) together with (23) states that each point c ($c = (X(\tau), \dot{X}(\tau), \tau)$), where $X, \dot{X}$ denote the respective solution functions for $x, \dot{x}$ of $\dot{x} \mathrel{\mathring{=}} e_{xt}$ satisfies

$$\mathrm{eval}'(\dot{x}, c) \in \mathrm{cl}(\mathrm{co}(\mathcal{L}(\tilde{e}_{xt}, c))) \qquad (28)$$

Here, $\mathrm{eval}'(\dot{x}, c)$ denotes $\dot{X}(\tau)$, and $\mathcal{L}(\tilde{e}_{xt}, c)$ can be simplified to $\mathcal{L}(\tilde{e}_{xt}, (X(\tau), \tau))$ since $\dot{x}$ does not occur in e_{xt}. We can then write (28) as $\dot{X}(\tau) \in \mathrm{cl}(\mathrm{co}(\mathcal{L}(f, (X(\tau), \tau))))$, where f denotes $\tilde{e}_{xt}$, which, under the assumption that f is continuous for t, denotes exactly the Filippov solution as defined by differential inclusion (2) in Section II.

VII. Example χ model

This section presents an example of a χ model. The example is the dry friction example treated in Section V

$$\begin{aligned}
&\langle\ x, v, F_{\mathrm{d}}, F_{\mathrm{f}} :: \mathsf{real},\ x = 0, v = 0\\
&|\ \ F_{\mathrm{d}} = f(t)\\
&,\quad \dot{x} = v\\
&,\ m\dot{v} = F_{\mathrm{d}} - F_{\mathrm{f}}\\
&,\ \ F_{\mathrm{f}} \mathrel{\mathring{=}} \mu F_N \,\mathrm{sign}(v)\\
&\rangle
\end{aligned}$$

where $x, v, F_{\mathrm{d}}, F_{\mathrm{f}}$ denote continuous variables, μ, F_N, m denote constants, t denotes the time, f denotes some function of time, and $x = 0, v = 0$ denote initial values, at time zero, for x and v.

VIII. Conclusions

Dynamical systems written in the form of differential algebraic equations with discontinuous right hand sides may result in ambiguities. From a mathematical point of view this need not be a problem, because a given equation can be interpreted – possibly in different ways – as a differential inclusion. This approach is not convenient for formal languages. In this paper we have introduced convex equations to model such systems in formal languages, such as the χ language [4]. The semantics of convex equations is formally defined.

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